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Local topological properties of differentiable mappings

by

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In this chapter we report some phenomena observed in a study of singularities of differentiable mappings, without proof. Proofs will be given in another place [F-1].

Let N and P be C^∞ manifolds with dim. $N = n$ and dim. $P = p$ respectively. Let $C^\infty(N,P)$ denote the space of all C^∞ mappings of N into P.

Then <u>there exists an infinite codimensional subset</u> Σ_∞ <u>of</u> $C^\infty(N,P)$ <u>such that</u>

I) <u>the topologically unstable singular points of an element</u> f <u>of</u> $C^\infty(N,P)$ <u>not belonging to</u> Σ_∞ <u>are isolated</u>,

II) <u>if</u> $n \leqq p$ <u>and if</u> $f \in C^\infty(N,P) - \Sigma_\infty$, <u>then every singularities of</u> f <u>is topologically equivalent to the cone of a topologically stable mapping of</u> S^{n-1} <u>into</u> S^{p-1}, <u>and</u>

III) <u>if the pair</u> (n,p) <u>of dimensions is in the nice range in J.Mather's sense, then</u> I) <u>and</u> II) <u>are still valid if we replace " topologically stable" by "</u> C^∞ <u>stable" and " topologically unstable " by "</u> C^∞ <u>unstable " respectively</u>.

Where the (open) cone of a mapping $g:X \to Y$ is the mapping $C(g):CX = X\times[0,1)/X\times\{0\} \to CY$ defined by $C(g)(x,t) = (g(x),t)$.

Also in the case of $n > p$, a phenomenon similar to II) is observed, which will appear in another place [F-2].

Note that II) asserts that if $n \leqq p$, then every mapping of N into P not belonging to the infinite codimensional subset Σ_∞ has locally a neat structure, in some cases a rather simple structure. For example

<u>Corollary. If</u> $n \geqq 3$ <u>and</u> $p \geqq 2n$, <u>then every mapping not belonging to</u> Σ_∞ <u>is a topological immersion. If</u> $n = 2$ <u>and</u> $p = 4$, <u>then the singularities of</u> $f \notin \Sigma_\infty$ <u>are topologically equivalent to cones of knots</u> $K^1 \subset S^3$, <u>e.t.c..</u>

In fact from II) and III), we see that if $p \geqq 2n$, then every singularity of a mapping of N into P not belonging to Σ_∞ is topologically equivalent to the cone of a C^∞ stable mapping of S^{n-1} into S^{p-1} which is a smooth embedding isotopic to the natural inclusion mapping $S^{n-1} \hookrightarrow S^{p-1}$ in the case $p \geqq 2n \geqq 6$, whose cone is the natural inclusion mapping $R^n \hookrightarrow R^p$. In the case $p = 2n = 4$, stable mappings of S^1 into S^3 are smooth knots $K^1 \hookrightarrow S^3$.

The proof given in [F-1] of the above result is essentially based on J.Mather's various stability theorems [M] and a transversality theorem given below.

We use the same notations as those of J.Mather's except for the followings. For integers r and s with $s > r \geqq 0$, $\pi^s_r : J^s(n,p) \to J^r(n,p)$ denotes the projection defined by $\pi^s_r(j^s f(0)) = j^r f(0)$. And for positive integers ℓ and m with $\ell \leqq m$, we put

$$\Delta_\ell = \{(j^k g_1(q_1),\ldots,j^k g_m(q_m)) \in {}_mJ^k(R^n,R^p) \mid g_1(q_1)=\ldots=g_\ell(q_\ell)\}.$$

Theorem. (A transversality.). Let W be a semi-algebraic subset of $J^r(n,p)$ and let X be a semi-algebraic submanifold of ${}_mJ^k(R^n,R^p)$. Then there exists a closed semi-algebraic subset Σ_W of $(\pi_r^{r+m(k+1)})^{-1}(W)$ having codimension $\geqq 1$ such that for any mapping $f:R^n \to R^p$ with $j^{r+m(k+1)}f(0) \in (\pi_r^{r+m(k+1)})^{-1}(W) - \Sigma_W$, there exists a neighborhood U of the origin of R^n such that

(i) ${}_mj^kf$ is transversal to X at every point of $(U - \{0\})^{(m)} = \{(q_1,\ldots,q_m) \in (U - \{0\})^m \mid q_i \neq q_j \text{ if } i \neq j\}$,

(ii) if X is of the form $X = (X_1 \times X_2 \times \ldots \times X_m) \cap \Delta_\ell$, where X_i are submanifolds of $J^k(R^n,R^p)$, then the pair $(\pi_1(({}_mj^kf)^{-1}(X) \cap (U - \{0\})^{(m)}), \{0\})$ satisfies Whitney's conditions (a) and (b), where $\pi_1:(R^n)^m \to R^n$ is the projection to the first factor: $\pi_1(q_1,\ldots,q_m) = q_1$.

Our transversality theorem has come out in an effort to understand the proof of Thom's theorem on topological sufficiency of map-germs, from which we have extracted our transversality theorem. And we can give another, more comprehensive, at least for the auther, proof of Thom,s theorem, which enables at the same time for us to estimate the order of topological sufficiency in Thom's theorem. see [F-1].

References.

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